Topological Modes in Dual Lattice Models

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Abstract

Lattice gauge theory with gauge group Z_P is reconsidered in four dimensions on a simplicial complex K. One finds that the dual theory, formulated on the dual block complex $\hat{K}$, contains topological modes which are in correspondence with the cohomology group $H^2(\hat{K}, Z_P)$, in addition to the usual dynamical link variables. This is a general phenomenon in all models with single plaquette based actions; the action of the dual theory becomes twisted with a field representing the above cohomology class. A similar observation is made about the dual version of the three dimensional Ising model. The importance of distinct topological sectors is confirmed numerically in the two dimensional Ising model where they are parameterized by $H^1(\hat{K}, Z_2)$.

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1 Introduction

The use of duality transformations in statistical systems has a long history, beginning with applications to the two dimensional Ising model [1]. Here one finds that the high and low temperature properties of the theory are related. This transformation has been extended to many other discrete models and is particularly useful when the symmetries involved are abelian; see [2] for an extensive review. All these studies have been confined to hypercubic lattices, or other regular structures, and these have rather limited global topological features.

Since lattice models are defined in a way which depends clearly on the connectivity of links or other regions, one expects some sort of topological effects generally. In this paper we will reconsider four dimensional Z_P lattice gauge theory based on actions which are functions of the holonomy around single plaquettes. The underlying lattice will be a simplicial complex which we take to model spacetime. To such a simplicial complex there is naturally associated a dual block complex which maps each k-simplex to a "dual block" of codimension k. This parallels the standard construction for hypercubic lattices only now the whole construction can be applied to spacetimes with more subtle topology. We find that the dual of Z_P lattice gauge theory in four dimensions, is another lattice gauge theory on the dual block complex, only now the link based dynamical variables are twisted with extra topological modes parameterized by the second cohomology group of the dual block complex with coefficients in Z_P. One might regard this as a coupling of normal lattice gauge theory with a topological field theory.

We begin by reviewing some important material on simplicial complexes, paying particular attention to some subtleties of the dual block construction. The dual transformation is then applied to four dimensional gauge theory as well as the Ising model in three dimensions. We also show explicitly on a two dimensional torus that the topological sectors in the dual Ising model are generally distinct.

2 Simplices and Dual Blocks

Here we will collect some standard definitions involving simplicial complexes and their dual block complexes; we refer to [3] for a complete exposition.

Let $V = \{v_1, ..., v_{N_0}\}$ be a collection of N_0 elements which we will call the *vertex set*. A *simplicial complex* K is a collection of finite nonempty subsets of V such that if $\sigma \in K$ so is every nonempty subset of σ. An element of K is called a *simplex* and its *dimension* is one less than the number of vertices it contains. We picture the 1-dimensional simplex $\{v_i, v_j\}$ as the line segment connecting two distinct points in Euclidean space. Similarly, $\{v_i, v_j, v_k\}$ can be pictured as the triangle with the three indicated vertices. An *orientation* of a simplex $\{v_0, ..., v_m\}$, denoted $[v_0, ..., v_m]$ is an equivalence class of the ordering of the vertices according to even and odd permutations; this gives a direction to line segments and a direction of circulation around the vertices of a triangle etc.

Let $K^{(m)} = \{\sigma_\alpha\}$ denote the collection of all oriented m-simplices in K. The group of $m-$*chains* on K, denoted by $C_m(K)$, is defined to be the set of all finite linear combinations $\sum_\alpha n_\alpha \, \sigma_\alpha$ with integer coefficients. A *boundary* operator $\partial_m : C_m(K) \to C_{m-1}(K)$ is defined by:

$$\partial_m[v_0, ..., v_m] = \sum_{i=0}^{m} (-1)^i \; [v_0, ..., \bar{v}_i, ..., v_m] \tag{1}$$

where we omit the vertex corresponding to $\bar{v}_i$. The $m-$*cochains* of K with coefficients in the abelian group G is $C^m(K, G) = Hom(C_m(K), G)$, and we have a *coboundary* operator $\delta^m : C^m(K, G) \to C^{m+1}(K, G)$ defined through the evaluation of an m-cochain c on an $(m + 1)$-simplex σ:

$$< \delta c, \sigma >=< c, \partial\sigma > \quad . \tag{2}$$

Let $Z^m(K, G)$ denote the kernel of δ^m and $B^m(K, G)$ the image of δ^{m-1}; the *cohomology group* $H^m(K, G)$ is then defined as the quotient Z^m/B^m.

We will be using the dual block complex associated to a given simplicial complex and this is defined in terms of a subdivision of the original complex. Given a geometric realization of a simplex, the central point is called the *barycenter*. The *barycentric subdivision* $Sd(K)$ of K is the new simplicial

complex obtained by subdividing every simplex in K at its barycenter. In this case, a 1-simplex becomes a union of two 1-simplices. Similarly a 2-simplex becomes divided into six 2-simplices; the new vertices are at the center of the original simplex as well as the center of each bounding 1-simplex. Given the simplicial complex K, the simplices of $Sd(K)$ are of the form $\pm[\hat{\sigma}_{i_1}, ...\hat{\sigma}_{i_m}]$, where $dim(\sigma_{i_1}) > \cdots > dim(\sigma_{i_m})$, and $\hat{\sigma}$ denotes the barycenter of σ. Associated to an m-simplex σ of K, we define the *dual block* $D(\sigma)$ to be the union of all open simplices of $Sd(K)$ for which $\hat{\sigma}$ is the final vertex. If K satisfies the additional technical conditions to be an n-dimensional PL-manifold [4], which we will assume, then it consists entirely of n-simplices and their faces, and the closed dual block, denoted by $\bar{D}(\sigma)$, has dimension $n-m$. One also has $\partial\bar{D}(\sigma) = \bar{D}(\sigma) - D(\sigma)$. The *dual block complex* $\hat{K}$ is the collection of all blocks which are dual to the simplices of K.

Let us discuss some examples to make the general setting a bit more concrete. Take the simplicial complex which represents S^4 given by the 4-simplices on the boundary of the 5-simplex $[0, 1, 2, 3, 4, 5]$. Here the complex contains, 6 vertices, 15 links, 20 2-simplices, and 15 3-simplices in addition to the 6 4-simplices. This case is exceptional in that the dual block complex is also a simplicial complex in its own right. It is not surprising since duality exchanges vertices for 4-blocks and links with 3-blocks and the number of simplices of these types match.

A more interesting example is given by a triangulation of CP^2 [5]. This complex has nine vertices, and is fully determined by specifying the 4-simplices which are 36 in number and these are listed in [5]. In this case, the dual block complex is not a simplicial complex and here it is useful to represent a given block by the blocks which lie on its boundary. To form the dual block complex, we begin by associating a 0-block (dual vertex) to each of the 4-simplices in K. The 1-blocks correspond to 4-simplices which share a common 3-face. The 2-blocks are slightly more difficult to enumerate, but one can make use of a theorem [3] which states that $\partial\bar{D}(\sigma)$ *is the union of all blocks* $D(\tau)$ *for which* τ *has* σ *as a proper face.* In this particular complex, 2-blocks contain from three to six dual links on their boundary. $\hat{K}$ is clearly not a simplicial complex but both K and $\hat{K}$ have subdivisions in common and encode the same topological information. One can proceed to enumerate all blocks with the help of the above theorem.

3 Dual Transformation

The partition function of Z_P lattice gauge theory is defined as a sum over all link variables $U_{ij} \in Z_P$ which we represent multiplicatively, and a Boltzmann weight factor for every 2-simplex in the simplicial complex K [6],

$$Z = \sum_{\{U_{ij}\}} \prod_{\Delta \in K^{(2)}} \exp[S(U_\Delta)] \quad . \tag{3}$$

The action S is a function of the holonomy U_Δ around Δ,

$$U_\Delta = U_{ij}\, U_{jk}\, U_{ki} \quad , \tag{4}$$

where the boundary of Δ is given by,

$$\partial\Delta = [j,k] - [i,k] + [i,j] \quad . \tag{5}$$

The character expansion of the Boltzmann weight is

$$\exp[S(U)] = \sum_{n=0}^{P-1} b_n\, U^n \quad , \tag{6}$$

where the b_n coefficients are the parameters of the theory. Usually one requires that the Boltzmann weight be insensitive to the orientations of the holonomies and this will restrict $b_{P-n} = b_n$. Introducing an integer $n_\Delta \in \{0, ..., P-1\}$ for each 2-simplex Δ, Z becomes,

$$Z = \sum_{\{U_{ij}\}} \prod_{\Delta \in K^{(2)}} \sum_{\{n_\Delta\}} b_{n_\Delta}\, U_\Delta^{n_\Delta} \quad . \tag{7}$$

The collection of the n_Δ for all 2-simplices in K is a 2-cochain. Now rearrange the order of factors, the idea being to collect all terms proportional to each link variable U_{ij}; we have

$$Z = \sum_{\{n_\Delta\}} \prod_{\Delta \in K^{(2)}} b_{n_\Delta} \prod_{[i,j] \in K^{(1)}} (\sum_{U_{ij}} (\prod_{\Delta \supset [i,j]} U_{ij}^{n_\Delta})) \quad , \tag{8}$$

where the last product in this equation is over all 2-simplices which contain the specified link $[i,j]$. Using the representation of a mod-P delta function,

$$\sum_{U \in Z_P} U^n = P\, \delta(n) \quad , \tag{9}$$

one obtains,

$$Z = P^{N_1} \sum_{\{n_\Delta\}} \prod_{\Delta \in K^{(2)}} b_{n_\Delta} \prod_{[i,j] \in K^{(1)}} \delta(\sum_{\Delta \supset [i,j]} n_\Delta) \ . \tag{10}$$

Notice that the sum in the delta function is over all 2-simplices which contain $[i,j]$ as a face. Now let us map each of the above quantities into the dual picture. First, each of the integer variables n_Δ associated to the 2-simplex Δ in $K^{(2)}$ becomes associated with the unique closed 2-block $\bar{D}(\Delta)$ in four dimensions;

$$n_{\bar{D}(\Delta)} = n_\Delta \ . \tag{11}$$

The utility of the dual transformation becomes apparent when one looks at the assembly of delta functions in (10). Each of these is associated to a link $[i,j]$ of the original simplicial complex; in the dual picture there is one delta function for each 3-block $\bar{D}([i,j])$. Moreover, one has that

$$\sum_{\Delta \supset [i,j]} n_\Delta = \sum_{\bar{D}(\Delta) \subset \partial \bar{D}([i,j])} n_{\bar{D}(\Delta)} \ . \tag{12}$$

To see this, we can appeal to Theorem 64.1 of [3] which states that $\partial \bar{D}(\sigma)$ *is the union of all blocks $D(\tau)$ for which τ has σ as a proper face*; in this case we take $\sigma = [i,j]$. So far, the above presentation parallels the usual case of a hypercubic lattice [2, 6], but now we see that we have the possibility of non-trivial solutions to the constraints. The last delta function says that the sum of the 2-cochains around the boundary of each 3-block must vanish mod-P; these are nothing but the conditions for a 2-cocycle with coefficients in Z_P on the dual block complex. To solve those constraints, we need the kernel of the coboundary operator,

$$\delta^2 : C^2(\hat{K}, Z_P) \to C^3(\hat{K}, Z_P) \tag{13}$$

operating on the 2-cochains of $\hat{K}$. The partition function expressed in terms of dual quantities is then

$$Z = P^{\hat{N}_3} \sum_{\{n_{\hat{\Delta}}\} \in Z^2(\hat{K}, Z_P)} \prod_{\hat{\Delta} \in \hat{K}^{(2)}} b_{n_{\hat{\Delta}}} \ , \tag{14}$$

where $\hat{K}^{(2)}$ is the set of all 2-blocks and $\hat{N}_m = N_{4-m}$ is the number of m-blocks. $Z^2(\hat{K}, Z_P)$ is nothing but the image of δ^1, the trivial 2-cocycles, together with the non-trivial cocycles which represent the cohomology classes:

$$Ker(\delta^2) = Im(\delta^1) \oplus H^2(\hat{K}, Z_P) \ . \tag{15}$$

Let us parameterize the solution by:

$$n_{\hat{\Delta}} = B_{\hat{\Delta}} + (\delta^1 A)_{\hat{\Delta}} \quad , \tag{16}$$

where B runs over the 2-cochains which cannot be written as the boundary of a link field as in the second term. Here A and B take their values in the additive group $\{0, ..., P-1\}$. We would like to write Z as a sum over A and B but we need to take account of the fact that δ^1 has a kernel; a simple sum over A would be an overcounting. To go a bit further, let us restrict the following discussion to the case where P is a prime number so Z_P is an algebraic field. The kernel of δ^1, the group of 1-cocycles, is parameterized by the image of δ^0 together with $H^1(\hat{K}, Z_P)$. When P is prime, the later cohomology group is then a sum of copies of Z_P; let h^1 denote the number of these copies, i.e. the dimension of $H^1(\hat{K}, Z_P)$ as a vector space over Z_P. The dimension of $Ker(\delta^1)$ is then,

$$\begin{aligned} dim \;\; Ker(\delta^1) &= h^1 + dim \;\; Im(\delta^0) \\ &= h^1 + (\hat{N}_0 - 1) \;\; . \end{aligned} \tag{17}$$

The expression for Z becomes,

$$Z = P^{(\hat{N}_3 - \hat{N}_0 - h^1 + 1)} \sum_{B \in H^2(\hat{K}, Z_P)} \sum_{\{A_{\hat{i}\hat{j}}\}} \prod_{\hat{\Delta} \in \hat{K}^{(2)}} b_{(B + \delta^1 A)_{\hat{\Delta}}} \quad . \tag{18}$$

In the dual theory, the new Boltzmann weight is therefore just proportional to $b_{(B+\delta^1 A)}$. In multiplicative Z_p notation,

$$U_{\hat{i}\hat{j}} = \exp[\frac{2\pi i}{P} \; A_{\hat{i}\hat{j}}] \quad , \quad W_{\hat{\Delta}} = \exp[\frac{2\pi i}{P} \; B_{\hat{\Delta}}] \quad , \tag{19}$$

and the dual action is then a function of the product

$$W_{\hat{\Delta}} \; U_{\hat{\Delta}} \quad . \tag{20}$$

Let us define the dual Boltzmann weight by,

$$\exp[\hat{S}(W_{\hat{\Delta}} \; U_{\hat{\Delta}})] = P^{(\hat{N}_3 - \hat{N}_0 - h^1 + 1)} \; b_{(B + \delta^1 A)_{\hat{\Delta}}} \quad , \tag{21}$$

and we have the final form of the partition function for four dimensional Z_P gauge theory, only now written in terms of the dual complex variables,

$$Z = \sum_{B \in H^2(\hat{K}, Z_P)} \sum_{\{U_{\hat{i}\hat{j}}\}} \prod_{\hat{\Delta} \in \hat{K}^{(2)}} \exp[\hat{S}(e^{\frac{2\pi i}{P} B_{\hat{\Delta}}} \; U_{\hat{\Delta}})] \quad . \tag{22}$$

We see then that the dual theory is generally another link based gauge theory with an action that depends on the usual holonomy of link fields but now twisted with an extra phase. This phase is topological in origin and one can consider the B field to be a topological excitation of a discrete topological field theory [7, 8, 9]. Since any lattice which captures the same topology will have the same number of these modes in the dual theory, it would be interesting to understand their role in the continuum limit. Both the original theory on K and the dual formulation on $\hat{K}$ have common subdivisions and in taking a continuum limit of K one is inducing a continuum limit in the dual picture. The extra modes captured in $H^2(\hat{K}, Z_P)$ are topological and, in a sense, already associated with the continuum limit. Perhaps the simplest example to study where these modes are present is CP^2 where $H^2(CP^2, Z_P) = Z_P$. We have remarked that this has a very simple simplicial presentation [5] and would be suitable for numerical studies.

It should not be surprising that the duality between the Ising model and Z_2 lattice gauge theory in three dimensions has the same extra set of topological modes. Consider the Ising model defined on a simplicial complex K where the partition function is given by

$$Z_I = \sum_{\{s_i\}} \prod_{[i,j]\in K^{(1)}} \exp[\beta\, s_i s_j] \quad . \tag{23}$$

Here one will make a character expansion by introducing an integer variable $n_{ij} \in \{0, 1\}$ for each link. Following the same procedure as before, one finds that the dual variables in three dimensions,

$$n_{\hat{\Delta}} = n_{\bar{D}([i,j])} = n_{ij} \tag{24}$$

are restricted to satisfy the 2-cocycle condition, and the 3d partition function Z_{I3} becomes identical in structure to (14),

$$Z_{I3} = 2^{\hat{N}_3} \sum_{\{n_{\hat{\Delta}}\}\in Z^2(\hat{K},Z_2)} \prod_{\hat{\Delta}\in\hat{K}^{(2)}} b_{n_{\hat{\Delta}}} \quad . \tag{25}$$

The subsequent steps are the same as in the previous case. Here we can be more explicit since we began with a definite action, and the final expression can be written as,

$$Z_{I3} = 2^{(\hat{N}_3-\hat{N}_0-h^1+1)} e^{\hat{\beta}_0\hat{N}_2} \sum_{B\in H^2(\hat{K},Z_2)} \sum_{\{U_{\hat{i}\hat{j}}\}} \prod_{\hat{\Delta}\in\hat{K}^{(2)}} \exp[\hat{\beta}\,(-1)^{B_{\hat{\Delta}}}\, U_{\hat{\Delta}}] \quad , \tag{26}$$

with the new parameters $\hat{\beta}$ and $\hat{\beta}_0$ given by,

$$\begin{aligned} \hat{\beta} &= \frac{1}{2}\ \ln[\coth\beta] \\ \hat{\beta}_0 &= \frac{1}{2}\ \ln[\cosh\beta\cdot\sinh\beta]\ \ . \end{aligned} \tag{27}$$

The dual theory is then a three dimensional Z_2 lattice gauge theory coupled with extra topological modes associated to $H^2(\hat{K}, Z_2)$. A simple space where these modes will appear is RP^3 which has $H^2(RP^3, Z_2) = Z_2$.

Similarly, an analysis of the two dimensional Potts model will yield a dual theory which depends on a topological mode belonging to $H^1(K, Z_P)$. In this case, one can see these modes on the torus with a square lattice. Let us look now at the two dimensional Ising model specified by the partition function (23). A square lattice with opposite sides identified has a dual lattice of precisely the same structure, and the number of vertices, links and squares is preserved by duality. In the dual picture we have in general,

$$Z_{I2} = 2^{\hat{N}_2 - 1} \sum_{B\in H^1(\hat{K},Z_2)} \sum_{\{A_i\}} \prod_{[\hat{i},\hat{j}]} b_{(B+\delta^0 A)_{\hat{i}\hat{j}}}\ \ . \tag{28}$$

The factor of 2^{-1} arises from the fact that $dim\, Ker(\delta^0) = 1$. Since the square lattice for T^2 and its dual are identical, the partition function can be written as,

$$Z_{I2} = 2^{N_0 - 1} e^{\hat{\beta}_0 N_1} \sum_{B\in H^1(K,Z_2)} \sum_{\{s_i\}} \prod_{[i,j]} \exp[\hat{\beta}\,(-1)^{B_{ij}}\, s_i\, s_j]\ \ , \tag{29}$$

with the new parameters $\hat{\beta}$ and $\hat{\beta}_0$ as in equation (27). The change in the parameters in the dual theory is the same as for the infinite square lattice [2, 6]; here the topology of the torus has been accounted for. A representation of the cohomology class $H^1(T^2, Z_2) = Z_2 \oplus Z_2$ is easy to find. This amounts to specifying the value of B on all links. Perhaps the simplest choice is to take B to be zero on all links except that the vertical links in the bottom row of squares are assigned x, and the horizontal links in the right column of squares are assigned y. The sum over H^1 amounts then to summing x and y over $\{0, 1\}$. Equation (29) has been checked numerically and one finds that the contributions from the topological sectors generally differ from each other.

The topological sectors in the dual two dimensional Ising model are presumably related to spin structures (boundary conditions) in the continuum limit, and this supports the view that they will be important in the other models we have considered. Similar sectors have been observed from an entirely different point of view [10] in the correlation functions of the Ising model on a cylinder.

Let us just remark that in a similar way, lattice models based upon the abelian group $U(1)$ will generally have topological modes in the dual theory which are parameterized by cohomology classes with integer coefficients.

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